

AN UNEXPECTED CONGRUENCE MODULO 5 FOR 4-COLORED GENERALIZED FROBENIUS PARTITIONS

JAMES A. SELLERS

On the occasion of the 125th anniversary of the birth of Srinivasa Ramanujan

ABSTRACT. In his 1984 AMS Memoir, George Andrews defined the family of k -colored generalized Frobenius partition functions. These are denoted by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. In that Memoir, Andrews proved (among many other things) that, for all $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod{5}$. Soon after, many authors proved congruence properties for various k -colored generalized Frobenius partition functions, typically with a small number of colors.

In 2011, Baruah and Sarmah proved a number of congruence properties for $c\phi_4$, all with moduli which are powers of 4. In this brief note, we add to the collection of congruences for $c\phi_4$ by proving this function satisfies an unexpected result modulo 5. The proof is elementary, relying on Baruah and Sarmah's results as well as work of Srinivasa Ramanujan.

1. INTRODUCTION

In his 1984 AMS Memoir, George Andrews [2] defined the family of k -colored generalized Frobenius partition functions which are denoted by $c\phi_k(n)$ where $k \geq 1$ is the number of colors in question. Among many things, Andrews [2, Corollary 10.1] proved that, for all $n \geq 0$, $c\phi_2(5n+3) \equiv 0 \pmod{5}$. Soon after, many authors proved similar congruence properties for various k -colored generalized Frobenius partition functions, typically for a small number of colors k . See, for example, [5, 6, 7, 9, 10, 11, 12, 13, 14].

In 2011, Baruah and Sarmah [3] proved a number of congruence properties for $c\phi_4$, all with moduli which are powers of 4. In this brief note, we add to the collection of congruences for $c\phi_4$ by proving this function satisfies an unexpected result modulo 5.

Theorem 1.1. *For all $n \geq 0$, $c\phi_4(10n+6) \equiv 0 \pmod{5}$.*

Our proof is elementary, relying on Baruah and Sarmah's results as well as work of Srinivasa Ramanujan.

2. AN ELEMENTARY PROOF OF THEOREM 1.1

Recall Ramanujan's functions

$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Date: February 26, 2013.

J. A. Sellers gratefully acknowledges the support of the Austrian American Educational Commission which supported him during the Summer Semester 2012 as a Fulbright Fellow at the Johannes Kepler University, Linz, Austria.

Using Jacobi's Triple Product Identity [4, Entry 19], we have the following well-known product representations for $\phi(q)$ and $\psi(q)$:

$$(2.1) \quad \phi(q) = \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2}$$

and

$$(2.2) \quad \psi(q) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}$$

Baruah and Sarmah [3, Theorem 2.1] proved the following valuable representation of the generating function for $c\phi_4$.

Theorem 2.1.

$$\sum_{n=0}^{\infty} c\phi_4(n)q^n = \frac{\phi^3(q^2) + 12q\phi(q^2)\psi^2(q^4)}{(q; q)_\infty^4}$$

where $(a; b)_\infty := (1-a)(1-ab)(1-ab^2)(1-ab^3)\dots$

From here, we wish to 2-dissect the generating function in Theorem 2.1 (because we want to study the coefficients of q^{10n+6} in the power series representation of the generating function for $c\phi_4(n)$). To complete this task, we follow the path laid out by Baruah and Sarmah [3]. We begin by rewriting the generating function in Theorem 2.1 as

$$\sum_{n=0}^{\infty} c\phi_4(n)q^n = \frac{\phi^3(q^2) + 12q\phi(q^2)\psi^2(q^4)}{(q; q^2)_\infty^4 (q^2; q^2)_\infty^4}.$$

Then we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-q)^n \\ &= \frac{\phi^3(q^2)}{(q^2; q^2)_\infty^4} \left\{ \frac{1}{(q; q^2)_\infty^4} + \frac{1}{(-q; q^2)_\infty^4} \right\} + 12q \frac{\phi(q^2)\psi^2(q^4)}{(q^2; q^2)_\infty^4} \left\{ \frac{1}{(q; q^2)_\infty^4} - \frac{1}{(-q; q^2)_\infty^4} \right\} \\ &= \frac{\phi^3(q^2)}{(q^2; q^2)_\infty^4 (q^2; q^4)_\infty^4} \{(-q; q^2)_\infty^4 + (q; q^2)_\infty^4\} + 12q \frac{\phi(q^2)\psi^2(q^4)}{(q^2; q^2)_\infty^4 (q^2; q^4)_\infty^4} \{(-q; q^2)_\infty^4 - (q; q^2)_\infty^4\}. \end{aligned}$$

As noted by Baruah and Sarmah [3, (3.11) and (3.12)], we can employ work of Ramanujan [4, Entry 25] to obtain

$$(-q; q^2)_\infty^4 + (q; q^2)_\infty^4 = 2 \frac{\phi^2(q^2)}{(q^2; q^2)_\infty^2}$$

and

$$(-q; q^2)_\infty^4 - (q; q^2)_\infty^4 = 8q \frac{\psi^2(q^4)}{(q^2; q^2)_\infty^2}.$$

These can be used in the above to obtain, after simplification,

$$\sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-q)^n = 2 \left\{ \frac{\phi^5(q^2)}{(q^2; q^2)_\infty^6 (q^2; q^4)_\infty^4} + 48q^2 \frac{\phi(q^2)\psi^4(q^4)}{(q^2; q^2)_\infty^6 (q^2; q^4)_\infty^4} \right\}.$$

Therefore,

$$(2.3) \quad \sum_{n \geq 0} c\phi_4(2n)q^n = \frac{\phi^5(q)}{(q; q)_\infty^6 (q; q^2)_\infty^4} + 48q \frac{\phi(q)\psi^4(q^2)}{(q; q)_\infty^6 (q; q^2)_\infty^4}.$$

We now utilize (2.1) and (2.2) in (2.3) to obtain

$$(2.4) \quad \sum_{n \geq 0} c\phi_4(2n)q^n = \frac{(q^2; q^2)_{\infty}^{29}}{(q; q)_{\infty}^{20}(q^4; q^4)_{\infty}^{10}} + 48q \frac{(q^2; q^2)_{\infty}^5 (q^4; q^4)_{\infty}^6}{(q; q)_{\infty}^{12}}$$

after elementary simplifications.

Our goal now is to prove that, when written as power series in q , each of the two functions on the right-hand side of (2.4) has the property that every coefficient of a term of the form q^{5n+3} is a multiple of 5. Then Theorem 1.1 follows.

With this in mind, we first note that

$$(2.5) \quad \frac{(q^2; q^2)_{\infty}^{29}}{(q; q)_{\infty}^{20}(q^4; q^4)_{\infty}^{10}} \equiv \frac{(q^{10}; q^{10})_{\infty}^5 (q^2; q^2)_{\infty}^4}{(q^5; q^5)_{\infty}^4 (q^{20}; q^{20})_{\infty}^2} \pmod{5}.$$

Now all of the functions of q^5 in the right-hand side of (2.5) can be ignored; in other words, if we can show that every coefficient of a term of the form q^{5n+3} is a multiple of 5 in the power series representation of the function $(q^2; q^2)_{\infty}^4$, then every coefficient of a term of the form q^{5n+3} is a multiple of 5 in the power series representation of the function

$$\frac{(q^{10}; q^{10})_{\infty}^5 (q^2; q^2)_{\infty}^4}{(q^5; q^5)_{\infty}^4 (q^{20}; q^{20})_{\infty}^2}$$

as well. So we focus our attention on $(q^2; q^2)_{\infty}^4$. We know from Euler's Pentagonal Number Theorem and a well-known result of Jacobi [1, page 11 and page 176] that

$$(2.6) \quad (q^2; q^2)_{\infty}^4 = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{k+m} (2k+1) q^{k(k+1)+m(3m-1)}.$$

We now consider exponents of q and want to know when

$$(2.7) \quad 5n+3 = k(k+1) + m(3m-1)$$

has a solution for integers k, m , and n . When we consider (2.7) mod 5, we have

$$3 \equiv k(k+1) + m(3m-1) \pmod{5}.$$

Completing the square gives

$$(2.8) \quad 0 \equiv 3(2k+1)^2 + (6m-1)^2 \pmod{5}.$$

We know that all squares are congruent to 0, 1, or 4 modulo 5. So we consider the nine possible cases in (2.8) and note rather quickly that $(2k+1)^2$ must be congruent to 0 modulo 5 (the other possibilities lead to no solutions). Thus, $2k+1$ is divisible by 5 whenever we consider the coefficients of the terms of the form q^{5n+3} . Therefore, by (2.6), we know that the coefficients of the terms of the form q^{5n+3} in (2.5) must be divisible by 5.

We turn our attention to the second term on the right-hand side of (2.4) and note that

$$(2.9) \quad \begin{aligned} 48q \frac{(q^2; q^2)_{\infty}^5 (q^4; q^4)_{\infty}^6}{(q; q)_{\infty}^{12}} &= 48q \frac{(q^2; q^2)_{\infty}^5 (q; q)_{\infty}^3 (q^4; q^4)_{\infty}^6}{(q; q)_{\infty}^{15}} \\ &\equiv 48q \frac{(q^{10}; q^{10})_{\infty}^5 (q^{20}; q^{20})_{\infty} (q; q)_{\infty}^3 (q^4; q^4)_{\infty}}{(q^5; q^5)_{\infty}^3} \pmod{5}. \end{aligned}$$

As before, we can ignore those functions which are functions of q^5 , so our goal is to focus on the coefficients of the terms of the form q^{5n+3} in

$$q(q; q)_\infty^3 (q^4; q^4)_\infty$$

which is equivalent to considering the coefficients of the terms of the form q^{5n+2} in

$$(q; q)_\infty^3 (q^4; q^4)_\infty.$$

As above, we know that

$$(2.10) \quad (q; q)_\infty^3 (q^4; q^4)_\infty = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{k+m} (2k+1) q^{k(k+1)/2 + 2m(3m-1)}.$$

Thus, we want to know when

$$(2.11) \quad 5n+2 = \frac{k(k+1)}{2} + 2m(3m-1)$$

has a solution for integers k, m , and n . When we consider (2.11) modulo 5, we have

$$(2.12) \quad 2 = \frac{k(k+1)}{2} + 2m(3m-1) \pmod{5}$$

and completing the square in (2.12) gives

$$(2.13) \quad 0 = (2k+1)^2 + 8(m-1)^2 \pmod{5}.$$

As above, we find that the only way that (2.13) can have solutions is if $2k+1 \equiv 0 \pmod{5}$. Thus, thanks to (2.10), we can conclude that all the coefficients of the terms of the form q^{5n+3} in

$$48q \frac{(q^2; q^2)_\infty^5 (q^4; q^4)_\infty^6}{(q; q)_\infty^{12}}$$

are also divisible by 5. This completes the proof of Theorem 1.1. \square

3. CONCLUDING THOUGHTS

We close with a number of thoughts. First, it is worth noting that Theorem 1.1 implies another congruence property modulo 5, this time for the function $\phi_4(n)$ (which is the number of generalized Frobenius partitions of n which allow up to 4 repetitions of an integer in either row). See Andrews [2] for more details. We can easily prove the following result:

Theorem 3.1. *For all $n \geq 0$, $\phi_4(10n+6) \equiv 0 \pmod{5}$.*

Proof. Garvan [6] proved that, for any prime p ,

$$\phi_{p-1}(n) \equiv c\phi_{p-1}(n) \pmod{p}$$

for any integer $n \geq 0$. Thus, Theorem 3.1 immediately follows from Theorem 1.1. \square

Secondly, we note that exactly the same kind of proof as that given for Theorem 1.1 can be employed to prove the following theorem:

Theorem 3.2. *For all $n \geq 0$, $\overline{c\phi}_4(10n+6) \equiv 0 \pmod{5}$.*

Note that the functions $\overline{c\phi}_k(n)$ were defined by Kolitsch [8] for any integer $k \geq 1$. Indeed, Baruah and Sarmah [3, (5.8)] prove that

$$\sum_{n=0}^{\infty} \overline{c\phi}_4(2n)q^n = 64q \frac{(q^4; q^4)_{\infty}^6}{(q^2; q^2)_{\infty}^7 (q; q^2)_{\infty}^{12}}.$$

As in the above work, we then see that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{c\phi}_4(2n)q^n &= 64q \frac{(q^4; q^4)_{\infty}^6 (q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^{12}} \\ (3.1) \qquad \qquad \qquad &\equiv 64q \frac{(q^{20}; q^{20})_{\infty} (q^4; q^4)_{\infty} (q^{10}; q^{10})_{\infty} (q; q)_{\infty}^3}{(q^5; q^5)_{\infty}^3} \pmod{5}. \end{aligned}$$

Thus, thanks to (3.1), we only need to consider the coefficients of q^{5n+2} in $(q^4; q^4)_{\infty} (q; q)_{\infty}^3$ and this was done in the latter part of the proof of Theorem 1.1 above.

REFERENCES

- [1] G. E. ANDREWS, *The Theory of Partitions*, The Encyclopedia of Mathematics and Its Applications Series, Addison-Wesley, NY, 1976
- [2] G. E. ANDREWS, *Generalized Frobenius Partitions*, Memoirs of the American Mathematical Society, **301**, Providence, RI, May 1984
- [3] N. D. BARUAH AND B. K. SARMAH, Congruences for generalized Frobenius partitions with 4 colors, *Discrete Math.* **311** (2011), 1892–1902
- [4] B. C. BERNDT, *Ramanujan's Notebooks, Part III*, Springer-Verlag, NY, 1991
- [5] D. EICHORN AND J. A. SELLERS, Computational Proofs of Congruences for 2- Colored Frobenius Partitions, *International Journal of Mathematics and Mathematical Sciences*, **29**, no. 6 (2002), 333–340
- [6] F. G. GARVAN, Generalizations of Dyson's Rank, PhD Thesis, Pennsylvania State University, 1986
- [7] L. W. KOLITSCH, An extension of a congruence by Andrews for generalized Frobenius partitions, *J. Comb. Theory. Ser. A* **45** (1987), 31–39
- [8] L. W. KOLITSCH, M -order generalized Frobenius partitions with M colors, *J. Num. Thy.* **39** (1990), 279–284
- [9] L. W. KOLITSCH, A Congruence for Generalized Frobenius Partitions with 3 Colors Modulo Powers of 3, in *Analytic Number Theory, Proceedings of a Conference in Honor of Paul T. Bateman*, Birkhauser Boston, Boston, MA, 1990, 343–348
- [10] J. LOVEJOY, Ramanujan-type congruences for three colored Frobenius partitions, *J. Num. Thy.* **85** (2) (2000), 283–290
- [11] K. ONO, Congruences for Frobenius partitions, *J. Num. Thy.* **57** (1) (1996), 170–180
- [12] P. PAULE AND C.-S. RADU, The Andrews-Sellers Family of Partition Congruences, *Advances in Mathematics* (2012), 819–838
- [13] J. A. SELLERS, Congruences Involving F-Partition Functions, *International Journal of Mathematics and Mathematical Sciences* **17** (1) (1994), 187–188
- [14] X. XIONG, Congruences modulo powers of 5 for three-colored Frobenius partitions, arXiv:1003.0072 [math NT], 2010

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA, SELLERSJ@PSU.EDU